Dark Matter problem set
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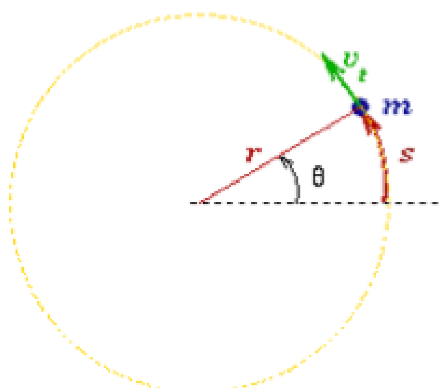
1. Rotation curves: In this problem set we will study rotational motion of point particles and galaxies:

a. Consider a point particle in circular motion with a fixed radius r (e.g a point on the rim of a wheel). We call the angle of a point on the wheel with respect to horizontal θ and the *angular velocity* of the point is then $\frac{d\theta}{dt} \equiv \omega$. What is the speed v and acceleration a of the point as a function of r and ω?



Figur 1: circular motion

b. Write an explicit parameterization of $\vec{x}(r,\theta)$ of the point particles position on the circle and derive your results in a) from this expression - (if you did not already do a) this way!)

c. Plot the speed $v(r)$ in a coordinate system as a function of r for fixed ω

d. Now consider a point particle of mass m (e.g. think of the Earth) gravitationally bound to a point particle of mass $M \gg m$ (e.g. think of the Sun). The particle m moves in a circular orbit with radius $r = R$ and with M at the center. Newtons gravitational constant is termed G. What is the speed of the point particle?

e. Plot the speed of the point particle in d) as a function of orbit radius r.

f. Now suppose $M = M(r)$ is actually a function of r (of course it cannot really be a point particle in this case...)! Take $M(r) = M_0 r^n$. Plot $v(r)$ for different values of n. For what value of n is the velocity of m simply constant as function of r?

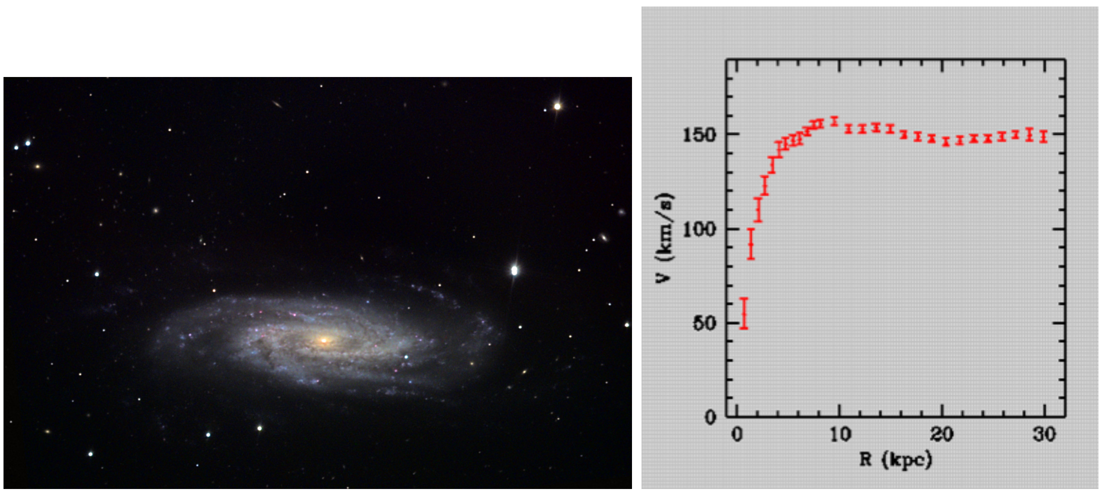
g. Now consider a spherical halo with radius R, total mass M and with uniform mass density

$\rho(r)$ which only depends on the radius r. Assume $\rho(r) = \rho_0 r^n$ where ρ_0 is a constant. What is ρ_0 as function of M and R? What is the total mass inside a certain radius $r < R$ as a function of n for $n = -2, 0, 2$?

h. What is the speed $v(r)$ of the particle orbiting in this extended spherical halo as a function of r, again for $n = -2, 0, 2$? - **hint: Newtons second theorem**.

i. Finally assume a point mass m orbits a heavier mass M (galactic center), which sits at the center of an extended spherical halo with uniform mass density $\rho(r) = \rho_0 r^n$ (dark matter halo), total mass of say $10M$ and radius R. Plot $v(r)$ for the values $n = -2, 0, 2$

j. Try to model the observed velocity curve of your favorite spiral galaxy using your formalism from above, e.g. the Galaxy rotation curve in the figure below.



Figur 2: NGC3198 Galaxy (Credit: John Vickery and Jim Matthes/Adam Block/NOAO/AURA/NSF). Right: Rotational velocity curve for the spiral galaxy NGC3198

k. Try to model the observed velocity curve of your favorite spiral galaxy using your formalism from above, e.g. the Galaxy rotation curve in the figure below.

Rotation curves grand finale: To be more precise, the visible part of a galaxy is (clearly) not quite a point mass and the DM halo density profiles is not quite (to the best of our knowledge) a simple *power-law* i.e. $\rho(r) = \rho_0 r^n$. So lets try to go state-of-the-art:

Radio-synthesis velocity maps of neutral hydrogen emission provide the most convincing evidence for darkmatter (e.g. Carignan and Freeman 1985). In NGC 3198 the rotation curve has been measured to 11h, or about 31kpc (van Albada et al. 1985) (See fig. 2 above).

Over such radii most galactic disks are slightly warped, and NGC 3198 is no exception. The warped disk is modeled as a set of concentric circular 75 76 CHAPTER 10. DARK MATTER IN DISK GALAXIES rings with inclinations varying fairly smoothly from 72^0 at the center to 76^0 at the edge. The fitted circular velocity of each ring, plotted against its radius, defines the rotation curve. After rising fairly slowly to a peak value of 157km=sec at about 3h, the rotation curve does not fall significantly below 150km=sec out to the last point measured. The rotation curve of NGC 3198 can be modeled by combining the gravitational forces of a thin exponential disk, and

- i) A thin *exponential disk*
- ii) A *dark halo* with a core and a *power-law* envelope

The thin exponential disk is parameterized by the density function

$$\Sigma(r) = \Sigma_0 \exp(-R/h) \tag{1}$$

The dark halo is parameterized by the density function

$$\rho(r)_h = \frac{\rho_0}{1 + (\frac{r}{a})^\alpha} \tag{2}$$

l. How do changing the parameters α and a affect the shape of the function $\rho(r)$?
m. The halo is assumed to be spherical. What is the velocity (squared) $v_h(r)^2$ as a function of radius with this profile if $\alpha = 2$ (In that case the halo profile is termed 'quasi-isothermal')?

The velocity from the visible dis parameterized by $\Sigma(r)$ gives a bit more complicated expression.

$$v_d^2(r) = 4\pi G \Sigma_0 h y^2 (I_0(\frac{r}{2h})K_0(\frac{r}{2h}) + I_1(\frac{r}{2h})K_1(\frac{r}{2h}))hhhhk \tag{3}$$

where $I_j(\frac{r}{2h}), K_j(\frac{r}{2h}$ are modified Bessel functions of the first and second kind (try to look them up and see if you can plot them using a math program?) The total circular velocity of as shown in the figure above is then as

$$v(r)_c^2 = \sqrt{v_h(r)^2 + v_d(r)^2} \tag{4}$$

n. Can you plot the $v(r)_c^2$ for $h = 2.68$ kpc and confirm it is a good description of the velocity curve for $1.9 < \alpha < 2.9$, $7 < a < 12$ kpc?

Questions for thought: The earth rotates around its axis. The earth rotates around the Sun. The solar system rotates around the center of the Galaxy. What are the speeds and accelerations in these motions? (try to look up e.g speeds and distances and infer

accelerations)?
Do we feel these accelerations?
If yes why, if no why not?
If you rotate a bucket fast, why does water stay in the bucket even when its upside down?
How fast do you have to rotate it for the water to stay inside ?
Suppose there was an ant instead of water inside the bucket and the ant could not see outside...at the top of the circle, in which direction would the ant *perceive* gravity?

Newtons 2 theorems (shell theorems)

First theorem: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.
Second theorem: The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shells matter were concentrated into a point at its center.
Note it follows that a point particle orbiting inside a spherical mass distribution experiences a gravitational force as if all the mass of the sphere *inside* the orbit was located at the center of the sphere.
A good discussion of the theorems can be found on Wikipedia here